Differential Geometry and its Applications 29 (2011) S234–S237

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

On the jets of vector bundle maps

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ARTICLE INFO

Article history:
Available online 30 April 2011
Communicated by I. Kolář

MSC:
58A20

Keywords:
Vector bundles
(q,r)-jets

ABSTRACT

We deduce an abstract characterization of (q,r)-jets of vector bundle maps.

1. Jets of vector bundle maps

Let (q,r) be a pair of non-negative integers such that $q \geqslant r$. We extend the concept of r-jets of manifold maps to the concept of (q,r)-jets of vector bundle maps.

Definition 1. Let $p_1 : E_1 \to M_1$ and $p_2 : E_2 \to M_2$ be vector bundles, $f_1, f_2 : E_1 \to E_2$ be vector bundle maps covering $\underline{f}_1, \underline{f}_2 : M_1 \to M_2$ and $x \in M$ be a point. Let (q,r) be a pair of non-negative integers such that $q \geqslant r$. We say that $j_x^{(q,r)} f_1 = j_x^{(q,r)} f_2$ iff

$$j_x^q(\underline{f}_1) = j_x^q(\underline{f}_2) \quad \text{and} \quad j_v^r(f_1) = j_v^r(f_2)$$

for some (and then for any) point $v \in E_{1x}$ from the fiber E_1 over x.

We have a smooth bundle

$$J^{(q,r)}(E_1, E_2) = \{ j_x^{(q,r)} f \mid f : E_1 \to E_2,\ x \in M_1 \}$$

over $M_1 \times M_2$. It is called the bundle of (q,r)-jets from E_1 into E_2.

Moreover, we have a well-defined composition of jets:

$$Y \circ X = j_x^{(q,r)}(g \circ f),$$

where $X = j_x^{(q,r)} f \in J_x^{(q,r)}(E_1, E_2)_y$ and $Y = j_y^{(q,r)} g \in J_y^{(q,r)}(E_2, E_3)_z$, $x \in M_1$, $y \in M_2$. Then for a vector bundle map $g_1 : E_2 \to E_4$ and a vector bundle local isomorphism $g_2 : E_1 \to E_3$ we have an induced map

$$J^{(q,r)}(g_1, g_2) : J^{(q,r)}(E_1, E_2) \to J^{(q,r)}(E_3, E_4)$$

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2011.04.046

given by

$$J^{(q,r)}(g_1,g_2)(X) = \left(j_y^{(q,r)} g_1\right) \circ X \circ \left(j_{g_2(x)}^{(q,r)}\left(g_2^{-1}\right)\right),$$

where $X \in J_x^{(q,r)}(E_1,E_2)_y$, $x \in M_1$, $y \in M_2$.

We have the product preserving property

$$J^{(q,r)}(E_1, E_2 \times E_3) = J^{(q,r)}(E_1,E_2) \times_{M_1} J^{(q,r)}(E_1,E_3).$$

2. An abstract characterization of (q,r)-jets

We are going to present an abstract viewpoint similar to [2] and [3].

Let $\mathcal{VB}$ be the category of all vector bundles and all vector bundle maps. We denote the set of all germs of $\mathcal{VB}$-morphisms of E_1 into E_2 by $G(E_1,E_2)$.

Consider a rule F transforming every pair (E_1,E_2) of $\mathcal{VB}$-objects into fibred manifold $F(E_1,E_2)$ over $M_1 \times M_2$ and a system of maps

$$\varphi_{E_1,E_2} : G(E_1,E_2) \to F(E_1,E_2)$$

commuting with projections of $G(E_1,E_2)$ and $F(E_1,E_2)$ onto $M_1 \times M_2$ for all (E_1,E_2). Clearly, if we interpret the construction of (q,r)-jets as an operation on germs

$$j_x^{(q,r)} f = j_x^{(q,r)}(germ_x f),$$

then $F = J^{(q,r)}$ and $\varphi = j^{(q,r)}$ is an example of such a pair.

We formulate the following axioms I–IV:

I. Every $\varphi_{E_1,E_2} : G(E_1,E_2) \to F(E_1,E_2)$ is surjective.
II. If $W_1, W_2 \in G_x(E_1,E_2)_y$ and $V_1, V_2 \in G_y(E_2,E_3)_z$, $\varphi(W_1) = \varphi(W_2)$ and $\varphi(V_1) = \varphi(V_2)$, then $\varphi(V_1 \circ W_1) = \varphi(V_2 \circ W_2)$.

By I and II, we have a well-defined composition

$$Y \circ X = \varphi(V \circ W)$$

for every $X = \varphi(W) \in F_x(E_1,E_2)_y$ and $Y = \varphi(V) \in F_y(E_2,E_3)_z$.

We write $\varphi_x f$ for $\varphi(germ_x f)$. For another pair of $\mathcal{VB}$-objects (E_3,E_4), every $\mathcal{VB}$-isomorphism $g_2 : E_1 \to E_3$ and every $\mathcal{VB}$-morphism $g_1 : E_2 \to E_4$ induce $F(g_2,g_1) : F(E_1,E_2) \to F(E_3,E_4)$ by

$$F(g_2,g_1)(X) = (\varphi_y g_1) \circ X \circ \varphi_{g_2(x)}\left(g_2^{-1}\right), \quad X \in F_x(E_1,E_2)_y,$$

where g_2^{-1} is constructed locally.

Moreover, we require

III. Each maps $F(g_2,g_1)$ is smooth.
IV. (Product property.) $F(E_1, E_2 \times E_3) = F(E_1,E_2) \times_{M_1} F(E_1,E_3)$.

Definition 2. A pair (F,φ) satisfying axioms I–IV will be called a jet-like homomorphism on germs of vector bundle maps.

Clearly, the pair $(F,\varphi) = (J^{(q,r)}, j^{(q,r)})$ is a jet-like homomorphism.

The main result of the present note is the following classification result.

Theorem 1. *Every jet-like homomorphism on germs of vector bundle maps is of the form* $(F,\varphi) = (J^{(q,r)}, j^{(q,r)})$.

The proof of Theorem 1 will occupy the rest of this note.

3. Proof of Theorem 1

A classical results reads that product preserving bundle functors on the category $\mathcal{M}f$ of all manifolds and all smooth maps coincide with the Weil functors T^A and natural transformations $\mu : T^C \to T^B$ are in bijection with the Weil algebras homomorphisms $\mu : C \to B$ [4]. Using [4] I. Kolář gave in [3] an abstract characterization of jet-like homomorphisms on germs of smooth maps between manifolds.

Similarly, the product preserving bundle functors on the category $\mathcal{FM}$ of all fibred manifolds and all smooth fibred maps (or on the category $\mathcal{F}ol$ of all foliated manifolds and all foliation respecting maps) are in bijection with the Weil algebra homomorphisms $\mu : A \to B$ between Weil algebras [6,8,1].

Using the results [6] or [8] M. Doupovec, I. Kolář and W. Mikulski gave in [2] an abstract characterization of jet-like homomorphisms on jet of fibred maps (or of foliation respecting maps).

In the proof of Theorem 1 we will use an analogous procedure as in the mentioned above papers [3] and [2]. Indeed, we will use a result saying that the product preserving gauge bundle functors H on the category $\mathcal{VB}$ of all vector bundles are in bijection with the pairs (A, V) of Weil algebras A and A-modules V with $\dim_{\mathbb{R}} V < \infty$ [7,9,5].

More precisely, we have two vector bundles $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $pt_{\mathbb{R}} : \mathbb{R} \to pt$, where pt is a one point manifold. Applying the gauge bundle functor H we obtain $A = H(id_{\mathbb{R}})$ and $V = H(pt_{\mathbb{R}})$. Conversely, we have the product preserving gauge bundle functor $T^{(A,V)} : \mathcal{VB} \to \mathcal{FM}$ defined by $T^{(A,V)}E = T^A E \otimes_A V$ (the A-tensor product of the A-module bundle $T^A p : T^A E \to T^A M$ with the A-module V) and $T^{(A,V)} f = T^A f \otimes_A id_V$.

Consider the trivial vector bundle $\mathbb{R}^{k,l} := (\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k)$. For a jet-like homomorphism (F, φ), we define $F_{k,l} E = F_0(\mathbb{R}^{k,l}, E)$ and $F_{k,l} f : F_{k,l} E \to F_{k,l} \overline{E}$, $F_{k,l} f(\varphi_0(g)) = \varphi_0(f \circ g)$. Then $F_{k,l}$ is a product preserving gauge bundle functor on $\mathcal{VB}$ corresponding to some $(A_{k,l}, V_{k,l})$. Every smooth fiber linear map $g : \mathbb{R}^{k,l} \to \mathbb{R}$ can be interpreted as an $\mathcal{VB}$-map $\tilde{g} : \mathbb{R}^{k,l} \to pt_{\mathbb{R}}$. Clearly in our situation $V_{k,l}$ is the factor-module of the $\mathcal{E}(\mathbb{R}^k)$-module $\mathcal{E}_{lin}(\mathbb{R}^{k,l})$ of germs of fiber linear maps on $\mathbb{R}^{k,l}$ at 0 with respect to the sub-module $\mathcal{N}_{k,l}$ of all $germ_0(g)$ satisfying $\varphi_0(\tilde{g}) = \varphi_0(\tilde{\nu})$, where ν is the zero function on $\mathbb{R}^{k,l}$ and $\mathcal{E}(\mathbb{R}^k)$ is the algebra of germs at 0 of smooth maps $\mathbb{R}^k \to \mathbb{R}$.

Write $G_0(\mathbb{R}^{\bar{k},\bar{l}}, \mathbb{R}^{k,l})_0$ for the set of all germs at 0 of $\mathcal{VB}$-maps $\mathbb{R}^{\bar{k},\bar{l}} \to \mathbb{R}^{k,l}$ satisfying $f(0) = 0$. By construction, submodules $\mathcal{N}_{k,l}$ have the following substitution property:

$(*)$ $X \in \mathcal{N}_{k,l}$ and $S \in G_0(\mathbb{R}^{\bar{k},\bar{l}}, \mathbb{R}^{k,l})_0$ implies $X \circ S \in \mathcal{N}_{\bar{k},\bar{l}}$.

Denote by $\mathcal{M}$ the maximal ideal in $\mathcal{E}(\mathbb{R}^k)$.

In the case $(F, \varphi) = (J^{(q,r)}, j^{(q,r)})$ we write $A^{(q,r)}_{k,l}$ and $V^{(q,r)}_{k,l}$.

Lemma 1. *We have $V^{(q,r)}_{k,l} = \mathcal{E}_{lin}(\mathbb{R}^{k,l})/\mathcal{N}^{(q,r)}_{k,l}$, where*

$(**)$ $\mathcal{N}^{(q,r)}_{k,l} = \mathcal{M}^{r+1}\mathcal{E}_{lin}(\mathbb{R}^{k,l})$ *and* $A^{(q,r)}_{k,l} = \mathcal{E}(\mathbb{R}^k)/\mathcal{M}^{q+1}$.

Proof. This is a simple observation. □

Given k and l, we say that a submodule $\mathcal{N} \subset \mathcal{E}_{lin}(\mathbb{R}^{k,l})$ has the substitution property, if condition $(*)$ holds for every $X \in \mathcal{N}$ and $S \in G_0(\mathbb{R}^{\bar{k},\bar{l}}, \mathbb{R}^{k,l})_0$.

Proposition 1. *Let $\mathcal{N} \subset \mathcal{E}_{lin}(\mathbb{R}^{k,l})$ be a finite codimension submodule with the substitution property $(*)$. Then there exists a non-negative integer r such that $\mathcal{N} = \mathcal{N}^{(q,r)}_{k,l}$ for any $q \geqslant r$.*

Proof. Let $x^1, \ldots, x^k, y^1, \ldots, y^l$ be the usual coordinates on $\mathbb{R}^{k,l}$.

Since $\mathcal{N}$ is of a finite codimension, there exists minimal r such that $germ_0((x^1)^{q+1} y^1) \in \mathcal{N}$. By the substitution property $(*)$ we have $\mathcal{N}^{(q,r)}_{k,l} \subset \mathcal{N}$.

By the minimality of r and the substitution property $(*)$ we can obtain the converse inclusion by a contradiction. □

In other words, for every k, l there exists $r(k,l)$ such that $\mathcal{N}_{k,l}$ is of the form $(**)$ for any $q \geqslant r(k,l)$.

Lemma 2. *The number $r = r(k,l)$ is independent of k and l.*

Proof. By the proof of Proposition 1, the number $r(k,l)$ depends on x^1 and y^1 only. If we have another $\mathcal{N}_{\bar{k},\bar{l}}$, then the substitution property $(*)$ implies our claim. □

Hence the $A_{k,l}$-module $V_{k,l}$ of $F_{k,l}$ is independent of k, l. Similarly, the Weil algebra $A_{k,l} = \mathcal{E}(\mathbb{R}^k)/\mathcal{M}^{q+1}$ for an integer $q \geqslant r$ is also independent of k, l. In fact, this is the manifold results from [3]. The inequality $q \geqslant r$ is a simple consequence of the fact that the $A_{k,l}$-module structure on $V_{k,l}$ is well-defined. Thus we have deduced the module version of the formula

$$F_{k,l} E = J^{q,r}_0\big(\mathbb{R}^{k,l}, E\big).$$

For a vector bundle $\bar{p} : \overline{E} \to \overline{M}$ with $\dim(\overline{E}) = k + l$, $\dim(\overline{M}) = k$ we define

$$P^{(q,r)}\overline{E} = inv\, J_0^{(q,r)}\big(\mathbb{R}^{k,l}, \overline{E}\big),$$

where *inv* indicates that we consider (q,r)-jets of local $\mathcal{VB}$-isomorphisms. This is a principal bundle over $\overline{M}$ with the structure group

$$G_{k,l}^{(q,r)} = inv\, J_0^{(q,r)}\big(\mathbb{R}^{k,l}, \mathbb{R}^{k,l}\big)_0.$$

Then one can directly verify that $F(\overline{E}, E)$ coincides with the associated bundle

$$F(\overline{E}, E) = P^{(q,r)}\overline{E}\big[J_0^{(q,r)}\big(\mathbb{R}^{k,l}, E\big)\big].$$

Indeed, for $X = germ_x f \in G_{\bar{x}}(\overline{E}, E)_x$, we take an arbitrary $u = j_0^{(q,r)}\psi \in P_{\bar{x}}^{(q,r)}\overline{E}$ and we put

$$\varphi(X) = \big\{j_0^{(q,r)}\psi, j_0^{(q,r)}(f \circ \psi)\big\}.$$

One verifies directly that this definition is independent of the choice of u.

This implies that $(F, \varphi) = (J^{(q,r)}, j^{(q,r)})$.

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